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Hamilton–Jacobi Treatment of Constrained Systems with Second-Order Lagrangians

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A new approach is examined in this paper for solving mechanical problems for both constrained and unconstrained systems with second-order Lagrangians, using the Hamilton–Jacobi formulation. The relevant Hamilton–Jacobi function is constructed first. This is then used to determine the solutions of the equations of motion for both systems.

KEY WORDS: constrained systems; Hamilton-Jacoby function; second-order Lagrangians.

1. INTRODUCTION

The study of singular systems has reached a great status in physics starting from the early development by Dirac (1950; 1964) of the generalized Hamiltonian formulation. Since then this formalism has found a wide range of application in field theory (Hanson, 1976; Sundermeyer, 1982; Gitman and Tyutin, 1990). In particular, the treatment of constrained systems with higher-order Lagrangians has been applied in many physical problems. Podolsky electrodynamics (Podolsky and Schwed, 1948), string theory (Polyakov, 1986), relativistic particles in general (Pisarski, 1986), and relativistic particles with curvature and torsion (Nesterenko, 1994) are only some examples.

Theories associated with higher-order regular Lagrangians were first developed by Ostrogradski (1850). These led to Euler's and Hamilton's equations of motion. Pons generalized Ostrogradski's theorem for singular Lagrangians to higher-order Lagrangians by extending Dirac's method (Pons, 1989). He also demonstrated the equivalence of Euler–Lagrange and Hamilton–Dirac equations for constrained systems derived from singular higher-order Lagrangians in the derivatives.

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The existence of constrained structure in higher-order systems was noticed by other authors (Battle *et al.*, 1988). Discerning this in second-order systems, these authors clarified the relation between the Hamiltonian and the Lagrangian constraints. They deduced the generalized Hamilton–Dirac equations in phase space. An important result of their work is that the primary Hamiltonian constraints, i.e., those which follow directly from the definition of the Legender transformations, come in two types: first-class constraints which have zero Poisson brackets with all other constraints; and second-class constraints which do not have this property. In addition, the operators relating Hamiltonian and the Lagrangian constraints were derived. These authors also extended their work to higher-order singular Lagrangians.

The Hamiltonian formalism for systems with singular Lagrangians of the second–order was constructed by Nesterenko (1989). He proposed a new method for obtaining the equations of motion in phase space for theories with singular Lagrangians by differentiation of the canonical Hamiltonian. The relation between the Hamiltonian and the Lagrangian was obtained. He assigned two kinds of constraints: primary and secondary. An important result of his work is the derivation of the secondary constraints in the framework of the Lagrangian formalism by differentiation of the Lagrangian constraints with respect to time.

The Hamilton–Jacobi formalism with the canonical approach for second-order singular Lagrangians was developed by invoking Caratheodory's (Caratheodory, 1967) equivalence Lagrangian method (Pimentel and Teixeira, 1996). The structure of the constraints and the existence of the primary constraints in second-order systems were discussed. In this approach, the equations of motion for the canonical variables of singular second-order systems were obtained as total differential equations in many variables, and the set of Hamilton–Jacobi partial differential equations (HJPDEs) for second-order singular systems was written for these systems. The generalization of the Hamilton–Jacobi formalism to higher-order singular Lagrangians was then examined (Pimentel and Teixeira, 1998).

Recently, another approach for solving mechanical problems of constrained systems using the Hamilton–Jacobi formulation for first-order singular Lagrangians was examined (Rabei *et al.*, 2002, 2003). The Hamilton–Jacobi function was obtained in the same manner as for regular systems. This was then used to determine the solutions of the equations of motion for constrained systems.

In this paper, the Hamilton–Jacobi formulation of constrained dynamical systems with second-order Lagrangians is studied. A general form for the solution of HJPDEs of these systems is proposed. The Hamilton–Jacobi function in configuration space is obtained by solving these equations. This leads to an extension of the previous approach (Rabei *et al.*, 2002, 2004) for solving mechanical problems with second-order constrained and unconstrained Lagrangian systems.

The paper is organized as follows. In Section 2, the Hamilton–Jacobi formulation is reviewed briefly for both constrained and unconstrained systems. In Section 3, a generalized method is proposed for determining the Hamilton–Jacobi function for both systems; the equations of motion are then derived from this function. In Section 4, several illustrative examples are discussed. The work closes with some concluding remarks in Section 5.

2. HAMILTON–JACOBI FORMULATION FOR SECOND-ORDER LAGRANGIANS

The Lagrangian formulation of second-order theories requires the configuration space formed by N generalized coordinates q_i, $\dot{q}_i$ and $\ddot{q}_i$:

$$L(q_i, \dot{q}_i, \ddot{q}_i), \quad i = 1, \ldots, N \tag{2.1}$$

The corresponding Euler–Langrangian equations of motion are obtained from

$$S = \int L(q_i, \dot{q}_i, \ddot{q}_i)\, dt \tag{2.2}$$

using the Hamilton principle:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) = 0 \tag{2.3}$$

This is a system in N differential equations of fourth-order; so we need $4N$ initial conditions to solve it.

The Hamiltonian formulation for second-order derivatives, first developed by Ostrogradski (1850), treats the derivatives q_i and $\dot{q}_i$ as coordinates. The transformation from the Lagrangian to the Hamiltonian approach is achieved by introducing the generalized momenta p_i, π_i conjugate to the generalized coordinates $q_i, \dot{q}_i$, respectively:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) \tag{2.4}$$

$$\pi_i = \frac{\partial L}{\partial \ddot{q}_i} \tag{2.5}$$

then writing the accelerations $\ddot{q}_i$ as functions of the coordinates q and velocities $\dot{q}_i$ as well as of the momenta p_i and π_i [$\ddot{q}_i = f(q_i, \dot{q}_i, p_i, \pi_i)$]. The phase space will then be spanned by the canonical variables (q_i, p_i) and $(\dot{q}_i, \pi_i)$.

Introducing the canonical Hamiltonian

$$H_C \equiv p_i\dot{q}_i + \pi_i\ddot{q}_i \left|_{\ddot{q}_i = f_i} - L\right|_{\ddot{q}_i = f_i}$$

one can write the equations of motion of any function g of the canonical variables as

$$\dot{g} = \{g, H_C\} \tag{2.6}$$

However, this procedure is admissible only if the determinant of the Hessian matrix,

$$H_{ij} \equiv \left(\frac{\partial^2 L}{\partial \ddot{q}_i \partial \ddot{q}_j} \right), \quad i, j = 1, \ldots, N$$

does not vanish; otherwise it will not be possible to express all the accelerations $\ddot{q}_i$ as functions of the canonical variables, and there will be relations such as

$$\Phi_\alpha(q_i, p_i; \dot{q}_i, \pi_i) = 0, \quad \alpha = 1, \ldots, m < 2(N-1)$$

connecting the momenta variables. As a consequence, we will not be able to treat the canonical variables as an independent set; instead, we have to use a formalism specially developed to deal with the interdependent canonical variables, i.e., a formalism for constrained systems (Dirac, 1950, 1964).

Prior to this we will give a brief review of Caratheodory's (Caratheodory, 1967) equivalent Lagrangian method (Pimentel and Teixeira, 1996). Let us consider a Lagrangian $L(q_i, \dot{q}_i, \ddot{q}_i, t)$. One can obtain a completely equivalent Lagrangian by introducing

$$L' = L(q_i, \dot{q}_i, \ddot{q}_i, t) - \frac{dS(q_i, \dot{q}_i, t)}{dt} \tag{2.7}$$

such that the auxiliary function $S(q_i, \dot{q}_i, t)$ must satisfy

$$\frac{\partial S}{\partial t} = -H_0 \tag{2.8}$$

where H_0 is defined as the usual Hamiltonian:

$$H_0 = p_i \dot{q}_i + \pi_i \ddot{q}_i - L \tag{2.9}$$

$$P_i = \frac{\partial S}{\partial q_i} \tag{2.10}$$

$$\pi_i = \frac{\partial S}{\partial \dot{q}_i} \tag{2.11}$$

These are the fundamental equations of the equivalence Lagrangian method; Eq. (2.8) is the relevant Hamilton–Jacobi partial differential equation.

If the rank of the Hessian matrix

$$\frac{\partial^2 L}{\partial \ddot{q}_i \partial \ddot{q}_j} \tag{2.12}$$

is $N - R$, $R < N$, then the generalized momenta conjugate to the generalized coordinates $\dot{q}_i$ are defined as

$$\pi_a = \frac{\partial L}{\partial \ddot{q}_a}, \quad a = R + 1, \ldots, N \tag{2.13}$$

$$\pi_\alpha = \frac{\partial L}{\partial \ddot{q}_\alpha}, \quad \alpha = 1, \ldots, R \tag{2.14}$$

Since the rank of the Hessian matrix is $N - R$, one can solve Eq. (2.13) to obtain $N - R$ accelerations $\ddot{q}_a$ in terms of q_i, $\dot{q}_i$, π_a, and $\ddot{q}_\alpha$ as follows:

$$\ddot{q}_a = w_a(q_i, \dot{q}_i, \pi_a, \ddot{q}_\alpha) \tag{2.15}$$

Substituting Eq. (2.15) into (2.14), one gets

$$\pi_\alpha = \left.\frac{\partial L}{\partial \ddot{q}_\alpha}\right|_{\ddot{q}_a = w_a(q_i, \dot{q}_i, \pi_a, \ddot{q}_\alpha)} = -H_\alpha^\pi(q_i, \dot{q}_i, p_a, \pi_a) \tag{2.16}$$

We can obtain a similar expression for the momenta p_α:

$$p_\alpha = -H_\alpha^P(q_i, \dot{q}_i, p_a, \pi_a) \tag{2.17}$$

where

$$p_a = \frac{\partial L}{\partial \dot{q}_a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_a}\right) \tag{2.18a}$$

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_\alpha}\right) \tag{2.18b}$$

Eqs. (2.16) and (2.17) become

$$H_\alpha'^\pi(q_i, \dot{q}_i, p_i, \pi_i) = \pi_\alpha + H_\alpha^\pi = 0, \tag{2.19a}$$

$$\alpha = 1, \ldots, R$$

$$H_\alpha'^p(q_i, \dot{q}_i, p_i, \pi_i) = p_\alpha + H_\alpha^p = 0, \tag{2.19b}$$

which are called primary constraints (Dirac, 1950, 1964). These relations indicate that the generalized momenta p_α and π_α are not independent of p_a and π_a, which is a natural result of the singular nature of the Lagrangian. The Hamiltonian H_0 is then defined as

$$H_0 = p_a \dot{q}_a + \dot{q}_\alpha p_\alpha|_{p_\beta = -H_\beta^p} + \pi_a w_a + \ddot{q}_\alpha \pi_\alpha|_{p_\beta = -H_\beta^\pi} - L(q_i, \dot{q}_i, \ddot{q}_\alpha, \ddot{q}_{a = w_a}), \quad \beta = 1, \ldots, R, \quad a = R + 1, \ldots, N. \tag{2.20}$$

Defining the momentum P_0 as

$$P_0 = \frac{\partial S}{\partial t} \tag{2.21}$$

one can write the corresponding set of HJPDEs as (Pimentel and Teixeira, 1996)

$$H_0' = P_0 + H_0\left(t, q_\alpha, \dot{q}_\alpha, q_a, q_a; p_a = \frac{\partial S}{\partial \dot{q}_a};\ \pi_a = \frac{\partial S}{\partial \dot{q}_a}\right) = 0 \tag{2.22a}$$

$$H_\alpha'^p = p_\alpha + H_\alpha^p\left(t, q_\alpha, \dot{q}_\alpha, q_a, \dot{q}_a; p_a = \frac{\partial S}{\partial \dot{q}_a};\ \pi_a = \frac{\partial S}{\partial \dot{q}_a}\right) = 0 \tag{2.22b}$$

$$H_\alpha'^\pi = p_\alpha + H_\alpha^\pi\left(t, q_\alpha, \dot{q}_\alpha, q_a, \dot{q}_a; p_a = \frac{\partial S}{\partial \dot{q}_a};\ \pi_a = \frac{\partial S}{\partial \dot{q}_a}\right) = 0 \tag{2.22c}$$

The equations of motion are written as total differential equations in many variables as follows (Pimentel and Teixeira, 1996):

$$dq_a = \frac{\partial H_0'}{\partial p_a} dt + \frac{\partial H_\alpha'^P}{\partial p_a} dq_\alpha + \frac{\partial H_\alpha'^\pi}{\partial p_a} d\dot{q}_\alpha \tag{2.23a}$$

$$d\dot{q}_a = \frac{\partial H_0'}{\partial \pi_a} dt + \frac{\partial H_\alpha'^P}{\partial \pi_a} dq_\alpha + \frac{\partial H_\alpha'^\pi}{\partial \pi_a} d\dot{q}_\alpha \tag{2.23b}$$

$$dp_i = -\frac{\partial H_0'}{\partial q_i} dt - \frac{\partial H_\alpha'^P}{\partial q_i} dq_\alpha - \frac{\partial H_\alpha'^\pi}{\partial q_i} d\dot{q}_\alpha \tag{2.23c}$$

$$d\pi_i = -\frac{\partial H_0'}{\partial \dot{q}_i} dt - \frac{\partial H_\alpha'^P}{\partial \dot{q}_i} dq_\alpha - \frac{\partial H_\alpha'^\pi}{\partial \dot{q}_i} d\dot{q}_\alpha \tag{2.23d}$$

We note from Eqs. (2.23) that the existence of constraints reduce the number of the equations of motion.

Here $q_0 = t$. Then set of Eqs. (2.23) is integrable (Pimentel and Teixeira, 1996; Muslih and Guler, 1998), if and only if

$$dH_0' \equiv 0 \tag{2.24a}$$

$$dH_\alpha'^P \equiv 0 \tag{2.24b}$$

$$dH_\alpha'^\pi \equiv 0 \tag{2.24c}$$

or if it leads to new secondary constraints (Dirac, 1950, 1964). In other words, if conditions (2.24) are not satisfied identically, one may consider them as new constraints and then test for the consistency conditions; repeating this procedure, one may then be obtain a set of constraints.

Eqs. (2.23) can be solved to obtain the coordinates $q_a, \dot{q}_a$ and the momenta p_i, π_i as functions of $q_\alpha, \dot{q}_\alpha$, and t (Muslih, 2002). The canonical formulation leads to the set of canonical phase space coordinates as follows:

$$q_a \equiv q_a(t, q_\alpha, \dot{q}_\alpha);$$

$$\dot{q}_a \equiv \dot{q}_a(t, q_\alpha, \dot{q}_\alpha);$$

$$p_i \equiv p_a(t, q_\alpha, \dot{q}_\alpha); \quad a = R+1, \ldots, N$$
$$\pi_i \equiv \pi_a(t, q_\alpha, \dot{q}_\alpha). \quad \alpha = 1, \ldots, R \tag{2.25}$$

3. DETERMINING THE HAMILTON–JACOBI FUNCTION FOR SECOND-ORDER LAGRANGIANS

3.1. Unconstrained Systems

Under certain conditions it is possible to separate the variables in the Hamilton–Jacobi equations, and the solution can then always be reduced to quadratures (Goldstein, 1980; Arnold, 1989; Brack and Bhaduri, 1997). In practice, the Hamilton–Jacobi technique becomes a useful computational tool only when such a separation can be effected. In general, coordinates q_i and $\dot{q}_i$ are said to be separable in the Hamilton–Jacobi equations when Hamilton's principal function can be split into three additive parts: one that depends only on the coordinates q_i; a second that depends on the coordinates $\dot{q}_i$; and a third that is entirely independent of the coordinates q_i and $\dot{q}_i$. In the cases to which we apply the method of separation of variables, the Hamiltonian will be take to be time independent for mathematical convenience. If we then restrict our work to such Hamiltonians, the Hamilton–Jacobi equation for second-order unconstrained systems will be

$$\frac{\partial S(q_i, \dot{q}_i, t)}{\partial t} + H_0\left(q_i, \dot{q}_i, p_i = \frac{\partial S}{\partial q_i}, \pi_i = \frac{\partial S}{\partial \dot{q}_i}\right) = 0, \quad i = 1, \ldots, N \tag{3.1}$$

We shall first try to find a solution that can be written in a separable form:

$$S(q_i, \dot{q}_i, t) = W(q_i) + W'(\dot{q}_i) + f(t) \tag{3.2}$$

Substituting this in Eq. (3.1), we get

$$\frac{df}{dt} = -H_0\left(q_i, \dot{q}_i, p_i = \frac{\partial W}{\partial q_i}, \pi_i, = \frac{\partial W'}{\partial \dot{q}_i}\right) \tag{3.3}$$

The left-hand side depends only on t; whereas the right-hand side depends only on the coordinates q_i and $\dot{q}_i$. Therefore, each side must be equal to a constant independent of q, $\dot{q}_i$, and t. Let this constant be $-E'$. We then have

$$f(t) = -E't = -\sum_{i=1}^{N} E'_i t$$

where $E' = \sum_{i=1}^{N} E'_i$. The Hamilton–Jacobi function becomes

$$S(q_i, \dot{q}_i, t) = W(q_i, E) + W'(\dot{q}_i, E, E') - E't, \tag{3.4}$$

together with

$$H_0\left(q_i, \dot{q}_i, p_i = \frac{\partial W}{\partial q_i};\ \ \pi_i = \frac{\partial W'}{\partial \dot{q}_i}\right) = E' \tag{3.5}$$

This shows that for time-independent Hamiltonians we can always separate out the time. We can proceed further by the method of separation of variables only if Eq. (3.5) is similarly separable in q_i and $\dot{q}_i$; that is, if a solution can be written in the form

$$W = \sum_i [W_i(q_i, E_i) + W_i'(\dot{q}_i, E_i, E_i')], \quad i = 1, \ldots, N \tag{3.6}$$

Once we have found the Hamilton–Jacobi function S, the equations of motion can be obtained by using the following canonical transformations (Goldstein, 1980; Arnold, 1989):

$$\eta_i = \frac{\partial S}{\partial E_i'} \tag{3.7a}$$

$$\lambda_i = \frac{\partial S}{\partial E_i} \tag{3.7b}$$

$$p_i = \frac{\partial S}{\partial q_i} \tag{3.8a}$$

$$\pi_i = \frac{\partial S}{\partial \dot{q}_i} \tag{3.8b}$$

where η_i and λ_i are constants and can be determined from the initial conditions.

One can solve Eqs. (3.7) and (3.8) to get

$$\dot{q}_i = \dot{q}_i(\eta_i, E_i, E_i', t) \tag{3.9a}$$

$$q_i = q_i(\lambda_i, \eta_i, E_i, E_i', t) \tag{3.9b}$$

$$p_i = p_i(\lambda_i, \eta_i, E_i, E_i', t) \tag{3.9c}$$

$$\pi_i = \pi_i(\eta_i, E_i, E_i', t) \tag{3.9d}$$

3.2. Constrained Systems with Second-Order Lagrangians

In this case, instead of considering the Hamilton–Jacobi equation (3.1), we shall be dealing with a set of HJPDEs, Eqs. (2.22). If we have the same conditions for separable coordinates and follow the same procedure just discussed, we can extend this method to constrained systems. Moreover, because of the singular nature of the dynamical Lagrangians, we should split (the q_i and $\dot{q}_i$ coordinates of the system into those corresponding to independent momenta, q_a and $\dot{q}_a$, and others corresponding to dependent momenta, q_α and $\dot{q}_\alpha$). Thus, we can guess a general solution for Eqs. (2.22) in the form

$$S(q_a, q_a, \dot{q}_a, \dot{q}_\alpha, t) = f(t) + W_a(q_a, E_a) + W'_a(\dot{q}_a, E_a, E'_a) + f_\alpha(q_\alpha) + f'_\alpha(\dot{q}_\alpha) + A \tag{3.11}$$

where $f(t) = -E't$; $E' = \sum_{a=1}^{N-R} E'_a.E'_a$ are the $(N - R)$ constants of integration and A is some other constant; q_α and $\dot{q}_\alpha$ are treated as independent variables, just as the time t.

Here again the resulting equation for $f(t)$ has the solution $f(t) = -\sum_{a=1}^{N-R} E'_a t$, and the remaining functions $W_a(q_a, E_a)$, $W'_a(\dot{q}_a, E_a, E'_a)$, $f_\alpha(q_\alpha)$, and $f'_\alpha(\dot{q}_\alpha)$ are the time-independent Hamilton–Jacobi equations.

Once we have found the Hamilton–Jacobi function S, the equations of motion can be obtained in the manner of regular systems, using the following canonical transformation (Goldstein, 1980; Arnold, 1989):

$$\eta_a = \frac{\partial S}{\partial E'_a} \tag{3.12a}$$

$$\lambda_a = \frac{\partial S}{\partial E_a} \tag{3.12b}$$

$$p_i = \frac{\partial S}{\partial q_i} \tag{3.13a}$$

$$\pi_i = \frac{\partial S}{\partial \dot{q}_i} \tag{3.13b}$$

where η_a and λ_a are constants and can be determined from the initial conditions. The number of η_a is equal to the rank of the Hessian matrix, $N - R$; so is the number of λ_a.

Eqs. (3.12) and (3.13) can be solved to give

$$\dot{q}_a = \dot{q}_a(\eta_a, E_a, E'_a, \dot{q}_\alpha, t) \tag{3.14a}$$

$$q_a = q_a(\lambda_a, \eta_a, E_a, E'_a, q_\alpha, t) \tag{3.14b}$$

$$p_i = p_i(\lambda_a, \eta_a, E_a, E'_a, q_\alpha, t) \tag{3.15a}$$

$$\pi_i = \pi_i(\eta_a, E_a, E'_a, \dot{q}_\alpha, t) \tag{3.15b}$$

From the initial conditions, one can then determine the constants η_a and λ_b.

Two remarks are in order here. The first is that, if the Hamiltonian H_α^p does not depend on p_a and Hamiltonian H_α^π does not depend on π_a, the separation of variables will be straightforward. The second is that, if H_α^p depends on p_a and H_a^π depends on π_a, also if H_0 depends on q_α or $\dot{q}_\alpha$ or both, the separation of variables will not be achieved directly. In this case a suitable change of variables that combine (q_a, q_α) or $(\dot{q}_a, \dot{q}_\alpha)$ or both should be introduced. One can then redefine the Lagrangian in terms of the new variables and restart the problem.

To this end, further insight into the physical significance of $S(q_i, \dot{q}_i, t)$ is gained by an examination of its total time derivative

$$\frac{dS}{dt} = \frac{\partial S}{\partial q_a}\dot{q}_a + \frac{\partial S}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial S}{\partial \dot{q}_a}\ddot{q}_a + \frac{\partial S}{\partial \dot{q}_\alpha}\ddot{q}_\alpha + \frac{\partial S}{\partial t} \tag{3.16}$$

$$= p_a\dot{q}_a + p_\alpha\dot{q}_\alpha + \pi_a\ddot{q}_a + \pi_\alpha\ddot{q}_\alpha - H_0 = L \tag{3.17}$$

Thus, Hamilton's principal function differs from the time integral of the Lagrangian only by a constant:

$$S = \int L\,dt + \text{constant}. \tag{3.18}$$

In actual calculations, however, one cannot find S in teerms of time directly from this integral unless q_i, $\dot{q}_i$, p_i, and π_i are known as function of time.

4. ILLUSTRATIVE EXAMPLES

In this section, we discuss four examples: one for a regular Lagrangian, and three for different types of singular Lagrangians. The idea is to demonstrate how we can find solutions of HJPDEs for both constrained and unconstrained systems with second-order Lagrangians.

4.1. Second-Order Regular Lagrangian

We start with the following regular Lagrangian:

$$L = \frac{1}{2}(\ddot{q}^2 - \dot{q}^2) \tag{4.1}$$

This describes the one-dimensional motion of a black box in which a harmonic oscillator is hidden (a system of units is chosen such that the angular frequency of oscillations is 1) (Olga, 1997).

The corresponding generalized momenta (2.4) and (2.5) are

$$p = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}} = -\dot{q} - \dddot{q} \tag{4.2}$$

$$\pi = \frac{\partial L}{\partial \ddot{q}} = \ddot{q} \tag{4.3}$$

The Hamiltonian H_0 is calculated as

$$H_0 = p\dot{q} + \frac{1}{2}\pi^2 + \frac{1}{2}\dot{q}^2 \tag{4.4}$$

The corresponding set of HJPDEs, Eq. (3.1), reads

$$H_0' = P_0 + H_0 = \frac{\partial S}{\partial t} + \dot{q}\frac{\partial S}{\partial q} + \frac{1}{2}\left(\frac{\partial S}{\partial \dot{q}}\right)^2 + \frac{1}{2}\dot{q}^2 \equiv 0 \tag{4.5}$$

Substituting Eq. (3.4) into (4.5), we have

$$\frac{\partial f}{\partial t} + \dot{q}\frac{\partial W}{\partial q} + \frac{1}{2}\left(\frac{\partial W'}{\partial \dot{q}}\right)^2 + \frac{1}{2}\dot{q}^2 \equiv 0 \tag{4.6}$$

Since H_0 is time-independent, one can write $f(t) = -E't$. Eq. (4.6) can then be written as

$$-E' + \dot{q}\frac{\partial W}{\partial q} + \frac{1}{2}\left(\frac{\partial W'}{\partial \dot{q}}\right)^2 + \frac{1}{2}\dot{q}^2 \equiv 0 \tag{4.7}$$

We note from Eq. (4.7) that W depends only on q and W' depends only on $\dot{q}$. This means that

$$\frac{\partial W}{\partial q} = E \tag{4.8}$$

so that

$$W = qE$$

Substituting Eq. (4.8) into (4.7), we obtain

$$-E' + \dot{q}E + \frac{1}{2}\left(\frac{\partial W'}{\partial \dot{q}}\right)^2 + \frac{1}{2}\dot{q}^2 \equiv 0 \tag{4.9}$$

This equation leads to

$$W'(\dot{q}, E, E') = \int \sqrt{2E' + E^2 - (\dot{q} + E)^2}\, d\dot{q} \tag{4.10}$$

With these results, the Hamilton–Jacobi function becomes

$$S = -E't + qE + \int \sqrt{2E' + E^2 - (\dot{q} + E)^2}\, d\dot{q} + A \tag{4.11}$$

The solutions for the generalized coordinates can be obtained from the transformations (3.7):

$$\eta = \frac{\partial S}{\partial E'} = -t + \int \frac{d\dot{q}}{\sqrt{2E' + E^2 - (\dot{q} + E)^2}} \tag{4.12}$$

$$\lambda = \frac{\partial S}{\partial E} = q + \int \frac{[E - (\dot{q} + E)]}{\sqrt{2E' + E^2 - (\dot{q} + E)^2}}\, d\dot{q} \tag{4.13}$$

These two equations can be solved, respectively, to give

$$\dot{q} = \sqrt{2E' + E^2}\sin(\eta + t) - E \tag{4.14}$$

$$q = \lambda - E(\eta + t) - \sqrt{2E' + E^2}\cos(\eta + t) \tag{4.15}$$

The other half of the equations of motion can be determined by using Eqs. (3.8), after substituting the result for $\dot{q}$:

$$p = \frac{\partial S}{\partial q} = E \tag{4.16}$$

$$\pi = \frac{\partial S}{\partial \dot{q}} = \sqrt{2E' + E^2 - (\dot{q} + E)^2} = \sqrt{2E' + E^2}\cos(\eta + t) \tag{4.17}$$

One gets the same results using the Hamiltonian formalism.

4.2. Two Primary First-Class Constraints

We consider the following singular Lagrangian:

$$L = \frac{1}{2}\left(\ddot{q}_1^2 + \ddot{q}_2^2\right) + \dot{q}_3\ddot{q}_3 + \dot{q}_3 q_3 + q_2\dot{q}_2 \tag{4.18}$$

The corresponding generalized momenta, (2.18), (2.13), and (2.14), are

$$p_1 = -\dddot{q}_1 \tag{4.19a}$$

$$p_2 = q_2 - \dddot{q}_2 \tag{4.19b}$$

$$p_3 = q_3 = -H_3^p \tag{4.19c}$$

$$\pi_1 = \ddot{q}_1 \tag{4.19d}$$

$$\pi_2 = \ddot{q}_2 \tag{4.19e}$$

$$\pi_3 = \dot{q}_3 = -H_3^\pi \tag{4.19f}$$

Here the primary constraints are represented by Eqs. (4.19c) and (4.19f) that can be written as

$$H_3'^p = p_3 - q_3 = 0 \tag{4.20a}$$

$$H_3'^\pi = \pi_3 - \dot{q}_3 = 0 \tag{4.20b}$$

The Hamiltonian H_0 is calculated as

$$H_0 = p_1\dot{q}_1 + (p_2 - q_2)\dot{q}_2 + \frac{1}{2}\left(\pi_1^2 + \pi_2^2\right) \tag{4.21}$$

The corresponding set of HJPDEs, Eqs. (2.22), reads

$$H'_0 = P_0 + H_0 = \frac{\partial S}{\partial t} + \dot{q}_1 \frac{\partial S}{\partial q_1} + \dot{q}_2 \left(\frac{\partial S}{\partial q_2} - q_2 \right) + \frac{1}{2} \left(\frac{\partial S}{\partial \dot{q}_1} \right)^2 + \frac{1}{2} \left(\frac{\partial S}{\partial \dot{q}_2} \right)^2 = 0 \tag{4.22a}$$

$$H_3'^p = p_3 - q_3 = \frac{\partial S}{\partial q_3} - q_3 = 0 \tag{4.22b}$$

$$H_3'^\pi = \pi - \dot{q}_3 = \frac{\partial S}{\partial \dot{q}_3} - \dot{q}_3 = 0 \tag{4.22c}$$

However, the Poisson bracket of $H_3'^p$ and H'_0 is equal to zero; so is the Poisson bracket of $H_3'^\pi$ and H'_0. This means that there are no secondary constraints and the Poisson bracket of $H_3'^p$ and $H_3'^\pi$ is equal to zero. These, then, are first-class constraints (Dirac, 1950, 1964).

With Eq. (3.11), the Hamilton–Jacobi function S can be written as

$$S(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3, t) = f(t) + W_1(q_1, E_1) + W_2(q_2, E_2) + W'_1(\dot{q}_1, E_1, E'_1) + W'_2(\dot{q}_2, E_2, E'_2) + f_3(q_3) + f'_3(\dot{q}_3) + A \tag{4.23}$$

The coordinates q_3 and $\dot{q}_3$ are treated as independent variables, just as the time t. Since the Hamiltonian H_0 is time independent, one can write

$$f(t) = -(E'_1 + E'_2)t$$

Substituting S into Eq. (4.22a), we have

$$-E'_1 + \dot{q}_1 \frac{\partial W_1}{\partial q_1} + \frac{1}{2} \left(\frac{\partial W'_1}{\partial \dot{q}_1} \right)^2 - E'_2 + \dot{q}_2 \left(\frac{\partial W_2}{\partial q_2} - q_2 \right) + \frac{1}{2} \left(\frac{\partial W'_2}{\partial \dot{q}_2} \right)^2 = 0 \tag{4.24}$$

We note that W_1 depends only on q_1 and W_2 depends only on q_2. We can then write

$$\frac{\partial W_1}{\partial q_1} = E_1 \tag{4.25a}$$

so that

$$W_1 = E_1 q_1$$

and

$$\frac{\partial W_2}{\partial q_2} - q_2 = E_2 \tag{4.25b}$$

so that

$$W_2 = E_2 q_2 + \frac{1}{2} q_2^2$$

Substituting Eqs. (4.25) into (4.24), we have

$$-E_1' + \dot{q}_1 E_1 + \frac{1}{2}\left(\frac{\partial W_1'}{\partial \dot{q}_1}\right)^2 - E_2' + \dot{q}_2 E_2 + \frac{1}{2}\left(\frac{\partial W_2'}{\partial \dot{q}_2}\right)^2 = 0 \qquad (4.26)$$

Separation of variables in this equation yields

$$\frac{1}{2}\left(\frac{\partial W_1'}{\partial \dot{q}_1}\right)^2 + \dot{q}_1 E_1 - E_1' = 0$$

$$\frac{1}{2}\left(\frac{\partial W_2'}{\partial \dot{q}_2}\right)^2 + \dot{q}_2 E_2 - E_2' = 0$$

The solution of these equations can be determined as

$$W_1'(\dot{q}_1, E_1, E_1') = \int \sqrt{2E_1' - 2\dot{q}_1 E_1}\, d\dot{q}_1$$

$$W_2'(\dot{q}_2, E_2, E_2') = \int \sqrt{2E_2' - 2\dot{q}_2 E_2}\, d\dot{q}_2$$

Using Eq. (4.22b), one finds $f_3(q_3) = \frac{1}{2}q_3^2$; and using Eq. (4.22c), one finds $f_3'(\dot{q}_3) = \frac{1}{2}\dot{q}_3^2$.
With these results, the Hamilton–Jacobi function becomes

$$S = (-E_1' - E_2')t + q_1 E_1 + q_2 E_2 + \frac{1}{2}q_2^2 + \int \sqrt{2E_1' - 2\dot{q}_1 E_1}\, d\dot{q}_1 + \int \sqrt{2E_2' - 2\dot{q}_2 E_2}\, d\dot{q}_2 + \frac{1}{2}{q_3}^2 + \frac{1}{2}\dot{q}_3^2 + A \qquad (4.27)$$

The solutions for the generalized coordinates can be obtained from the transformations (3.12):

$$\eta_1 = \frac{\partial S}{\partial E_1'} = -t + \int \frac{d\dot{q}_1}{\sqrt{2E_1' - 2\dot{q}_1 E_1}} \qquad (4.28a)$$

$$\eta_2 = \frac{\partial S}{\partial E_2'} = -t + \int \frac{d\dot{q}_2}{\sqrt{2E_2' - 2\dot{q}_2 E_2}} \qquad (4.28b)$$

$$\lambda_1 = \frac{\partial S}{\partial E_1} = q_1 - \int \frac{\dot{q}_1 d\dot{q}_1}{\sqrt{2E_1' - 2\dot{q}_1 E_1}} \qquad (4.28c)$$

$$\lambda_2 = \frac{\partial S}{\partial E_2} = q_2 - \int \frac{\dot{q}_2 d\dot{q}_2}{\sqrt{2E_2' - 2\dot{q}_2 E_2}} \tag{4.28d}$$

Solving these four equations, one gets

$$\dot{q}_1 = \frac{E_1'}{E_1} - \frac{E_1}{2}(\eta_1 + t)^2 \tag{4.29a}$$

$$\dot{q}_2 = \frac{E_2'}{E_2} - \frac{E_2}{2}(\eta_2 + t)^2 \tag{4.29b}$$

$$q_1 = \lambda_1 + \frac{E_1'}{E_1}(\eta_1 + t) - \frac{E_1}{6}(\eta_1 + t)^3 \tag{4.29c}$$

$$q_2 = \lambda_2 + \frac{E_2'}{E_2}(\eta_2 + t) - \frac{E_2}{6}(\eta_2 + t)^3 \tag{4.29d}$$

The other half of the equations of motion can be determined by using Eqs. (3.13), after substituting the results for $\dot{q}_1$ and $\dot{q}_2$:

$$p_1 = \frac{\partial S}{\partial q_1} = E_1 \tag{4.30a}$$

$$p_2 = \frac{\partial S}{\partial q_2} = E_2 + q_2 = E_2 + \lambda_2 + \frac{E_2'}{E_2}(\eta_2 + t) - \frac{E_2}{6}(\eta_2 + t)^3 \tag{4.30b}$$

$$p_3 = \frac{\partial S}{\partial q_3} = q_3 \tag{4.30c}$$

$$\pi_1 = \frac{\partial S}{\partial \dot{q}_1} = \sqrt{2E_1' - 2\dot{q}_1 E_1} = -E_1(\eta_1 + t) \tag{4.30d}$$

$$\pi_2 = \frac{\partial S}{\partial \dot{q}_2} = \sqrt{2E_2' - 2\dot{q}_2 E_2} = -E_2(\eta_2 + t) \tag{4.30e}$$

$$\pi_3 = \frac{\partial S}{\partial \dot{q_3}} = \dot{q}_3 \tag{4.30f}$$

where q_3 and $\dot{q}_3$ are arbitrary parameters. One can show that these results are in exact agreement with those obtained using the canonical approach, Eqs. (2.23), as well as the Dirac approach.

4.3. Second-Class Constraints

We now consider the following singular Lagrangian:

$$L = \frac{1}{2}\left(\ddot{q}_1^2 + \ddot{q}_2^2\right) + \dot{q}_3\ddot{q}_3 - \frac{1}{2}\dot{q}_3^2 \tag{4.31}$$

The corresponding generalized momenta read

$$p_1 = -\dddot{q}_1 \tag{4.32a}$$

$$p_2 = -\dddot{q}_2 \tag{4.32b}$$

$$p_3 = -\dot{q}_3 = -H_3'^p \tag{4.32c}$$

$$\pi_1 = \ddot{q}_1 \tag{4.32d}$$

$$\pi_2 = \ddot{q}_2 \tag{4.32e}$$

$$\pi_3 = \dot{q}_3 = -H_3'^\pi \tag{4.32f}$$

The primary constraints are represented by Eqs. (4.32c) and (4.32f) that can be written as

$$H_3'^p = p_3 + \dot{q}_3 = 0 \tag{4.33a}$$

$$H_3'^\pi = \pi_3 - \dot{q}_3 = 0 \tag{4.33b}$$

The Hamiltonian H_0 is calculated as

$$H_0 = p_1\dot{q}_1 + p_2\dot{q}_2 + \frac{1}{2}\left(\pi_1^2 + \pi_2^2\right) - \frac{1}{2}\dot{q}_3^2 \tag{4.34}$$

The corresponding set of HJPDEs, (2.22), reads

$$H_0' = P_0 + H_0 = \frac{\partial S}{\partial t} + \dot{q}_1\frac{\partial S}{\partial q_1} + \dot{q}_2\frac{\partial S}{\partial q_2} + \frac{1}{2}\left(\frac{\partial S}{\partial \dot{q}_1}\right)^2 + \frac{1}{2}\left(\frac{\partial S}{\partial \dot{q}_2}\right)^2 - \frac{1}{2}\dot{q}_3^2 = 0 \tag{4.35a}$$

$$H_3'^p = p_3 + H_3^p = \frac{\partial S}{\partial q_3} + \dot{q}_3 = 0 \tag{4.35b}$$

$$H_3'^\pi = \pi_3 + H_3^\pi = \frac{\partial S}{\partial \dot{q}_3} - \dot{q}_3 = 0 \tag{4.35c}$$

However, the Poisson bracket of $H_3'^p$ and H_0' is equal to zero, and the Poisson bracket of $H_3'^\pi$ and H_0' is not identically zero; it gives a new (secondary) constraint (Dirac, 1950, 1964):

$$H_3''^\pi = \dot{q}_3 = 0 \tag{4.36}$$

There are no further constraints. Following the Dirac classification (Dirac, 1950, 1964), the constraints are of second-class.

Taking Eq. (4.36) into account, one can rewrite the primary constraint and the Hamiltonian, Eqs. (4.33) and (4.34), as

$$H_3'^p = p_3 = 0$$

$$H_3'^\pi = \pi_3 = 0$$

$$H_0 = p_1\dot{q}_1 + p_2\dot{q}_2 + \frac{1}{2}\left(\pi_1^2 + \pi_2^2\right)$$

Then the corresponding set of HJPDEs reads

$$H_0' = P_0 + H_0 = \frac{\partial S}{\partial t} + \dot{q}_1\frac{\partial S}{\partial q_1} + \dot{q}_2\frac{\partial S}{\partial q_2} + \frac{1}{2}\left(\frac{\partial S}{\partial \dot{q}_1}\right)^2 + \frac{1}{2}\left(\frac{\partial S}{\partial \dot{q}_2}\right)^2 = 0 \quad (4.37a)$$

$$H_3'^{p} = p_3 + H_3^{p} = \frac{\partial S}{\partial q_3} = 0 \quad (4.37b)$$

$$H_3'^{\pi} = \pi_3 + H_3'^{\pi} = \frac{\partial S}{\partial \dot{q}_3} = 0 \quad (4.37c)$$

The Hamilton–Jacobi function S can be determined by

$$S(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3, t) = f(t) + W_1(q_1, E_1) + W_2(q_2, E_2) + W_1'(\dot{q}_1, E_1, E_1') + W_2'(\dot{q}_2, E_2, E_2') + f_3(q_3) + f_3'(\dot{q}_3) + A \quad (4.38)$$

The coordinates q_3 and $\dot{q}_3$ are treated as independent variables, just as the time t. Since the Hamiltonian H_0 is time independent, one can write

$$f(t) = -(E_1' + E_2')t$$

Substituting Eqs. (4.38) into (4.37a), one can obtain

$$-E_1' + \dot{q}_1\frac{\partial W_1}{\partial q_1} + \frac{1}{2}\left(\frac{\partial W_1'}{\partial \dot{q}_1}\right)^2 - E_2' + \dot{q}_2\frac{\partial W_2}{\partial q_2} + \frac{1}{2}\left(\frac{\partial W_2'}{\partial \dot{q}_2}\right)^2 = 0 \quad (4.39)$$

From Eq. (4.39) we note that W_1 depends only on q_1 and W_2 depends only on q_2. We can then write

$$\frac{\partial W_1}{\partial q_1} = E_1 \quad (4.40a)$$

so that

$$W_1 = E_1 q_1$$

and

$$\frac{\partial W_2}{\partial q_2} = E_2 \quad (4.40b)$$

so that

$$W_2 = E_2 q_2$$

Substituting Eqs. (4.40) into (4.39), we have

$$-E_1' + \dot{q}_1 E_1 + \frac{1}{2}\left(\frac{\partial W_1'}{\partial \dot{q}_1}\right)^2 - E_3' + \dot{q}_3 E_3 + \frac{1}{2}\left(\frac{\partial W_3'}{\partial \dot{q}_3}\right)^2 = 0 \quad (4.41)$$

Separation of variables in this equation leads to

$$\frac{1}{2}\left(\frac{\partial W_1'}{\partial \dot{q}_1}\right)^2 + \dot{q}_1 E_1 - E_1' = 0$$

$$\frac{1}{2}\left(\frac{\partial W_2'}{\partial \dot{q}_2}\right)^2 + \dot{q}_2 E_2 - E_2' = 0$$

The solution of these two equations can be determined as

$$W_1'(\dot{q}_1, E_1, E_1') = \int \sqrt{2E_1' - 2\dot{q}_1 E_1}\, d\dot{q}_1$$

$$W_2'(\dot{q}_2, E, E_2') = \int \sqrt{2E_2' - 2\, E_2 \dot{q}_2}\, d\dot{q}_2$$

Using Eq. (4.37b), one finds $f_3(q_3)$ =constant; and using Eq. (4.37c), one finds $f_3'(\dot{q}_3) =$ constant.
With these result, the Hamilton–Jacobi function becomes

$$S = (-E_1' - E_2')t + q_1 E_1 + q_2 E_2 + \int \sqrt{2E_1' - 2\dot{q}_1 E_1}\, d\dot{q}_1 + \int \sqrt{2E_2' - 2\dot{q}_2 E_2}\, d\dot{q}_2 + A. \quad (4.42)$$

The solution for the generalized coordinates can be obtained from the transformations (3.12):

$$\eta_1 = \frac{\partial S}{\partial E_1'} = -t + \int \frac{d\dot{q}_1}{\sqrt{2E_1' - 2\dot{q}_1 E_1}} \quad (4.43a)$$

$$\eta_2 = \frac{\partial S}{\partial E_2'} = -t + \int \frac{d\dot{q}_2}{\sqrt{2E_2' - 2\dot{q}_2 E_2}} \quad (4.43b)$$

$$\lambda_1 = \frac{\partial S}{\partial E_1} = q_1 - \int \frac{\dot{q}_1 d\dot{q}_1}{\sqrt{2E_1' - 2\dot{q}_1 E_1}} \quad (4.43c)$$

$$\lambda_2 = \frac{\partial S}{\partial E_2} = q_2 - \int \frac{\dot{q}_2 d\dot{q}_2}{2E_2' - 2\dot{q}_2 E_2} \quad (4.43d)$$

These equations can be solved, respectively, to give

$$\dot{q}_1 = \frac{E_1'}{E_1} - \frac{E_1}{2}(\eta_1 + t)^2 \quad (4.44a)$$

$$\dot{q}_2 = \frac{E_2'}{E_2} - \frac{E_2}{2}(\eta_2 + t)^2 \quad (4.44b)$$

$$q_1 = \lambda_1 + \frac{E_1'}{E_1}(\eta_1 + t) - \frac{E_1}{6}(\eta_1 + t)^3 \tag{4.44c}$$

$$q_2 = \lambda_2 + \frac{E_2'}{E_2}(\eta_2 + t) - \frac{E_2}{6}(\eta_2 + t)^3 \tag{4.44d}$$

The other half of the equation of motion can be determined by using Eqs. (3.13), after substituting the results for $\dot{q}_1$ and $\dot{q}_3$:

$$p_1 = \frac{\partial S}{\partial q_1} = E_1 \tag{4.45a}$$

$$p_2 = \frac{\partial S}{\partial q_2} = E_2 \tag{4.45b}$$

$$p_3 = \frac{\partial S}{\partial q_3} = 0 \tag{4.45c}$$

$$\pi_1 = \frac{\partial S}{\partial \dot{q}_1} = \sqrt{2E_1' - 2\dot{q}_1 E_1} = -E_1(\eta_1 + t) \tag{4.45d}$$

$$\pi_2 = \frac{\partial S}{\partial \dot{q}_2} = \sqrt{2E_2' - 2\dot{q}_2 E_2} = -E_2(\eta_2 + t) \tag{4.45e}$$

$$\pi_3 = \frac{\partial S}{\partial \dot{q}_3} = 0 \tag{4.45f}$$

These results are in exact agreement with those obtained using the canonical approach, Eqs. (2.23), as well as the Dirac approach.

4.4. First and Second-Class Constraints

The final example has been constructed so as to contain first as well as second-class constraints:

$$L = \frac{1}{2}(\ddot{q}_1^2 + \ddot{q}_2^2) - \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{2}q_3^2 + \dot{q}_3\ddot{q}_3 \tag{4.46}$$

The corresponding generalized momenta read

$$p_1 = -\dot{q}_1 - \dddot{q}_1 \tag{4.47a}$$

$$p_2 = -\dot{q}_2 - \dddot{q}_2 \tag{4.47b}$$

$$p_3 = 0 = -H_3^p \tag{4.47c}$$

$$\pi_1 = \ddot{q}_1 \tag{4.47d}$$

$$\pi_2 = \ddot{q}_2 \tag{4.47e}$$

$$\pi_3 = \dot{q}_3 = -H_3^\pi \tag{4.47f}$$

Eqs. (4.47c) and (4.47f) represent primary constraints that can be written as

$$H_3'^p = p_3 = 0 \tag{4.48a}$$

$$H_3'^\pi = \pi_3 - \dot{q}_3 = 0 \tag{4.48b}$$

The Hamiltonian H_0 is calculated as

$$H_0 = p_1\dot{q}_1 + p_2\dot{q}_2 + \frac{1}{2}\left(\pi_1^2 + \pi_2^2\right) + \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) - \frac{1}{2}q_3^2 \tag{4.49}$$

The corresponding set of HJPDEs, (2.22), reads

$$H_0' = P_0 + H_0 = \frac{\partial S}{\partial t} + \dot{q}_1\frac{\partial S}{\partial q_1} + \dot{q}_2\frac{\partial S}{\partial q_2} + \frac{1}{2}\left(\frac{\partial S}{\partial \dot{q}_1}\right)^2 + \frac{1}{2}\left(\frac{\partial S}{\partial \dot{q}_2}\right)^2 + \frac{1}{2}\dot{q}_1^2 + \frac{1}{2}\dot{q}_2^2 - \frac{1}{2}q_3^2 = 0 \tag{4.50d}$$

$$H_3'^p = p_3 = \frac{\partial S}{\partial q_3} = 0 \tag{4.50e}$$

$$H_3'^\pi = \pi - \dot{q}_3 = \frac{\partial S}{\partial \dot{q}_3} - \dot{q}_3 = 0 \tag{4.50f}$$

However, the Poisson bracket of $H_3'^p$ and H_0' is not identically zero; it gives a new (secondary) constraint (Dirac, 1950, 1964):

$$H_3''^p = q_3 = 0 \tag{4.51}$$

There are no further constraints. Following the Dirac classification (Dirac, 1950, 1964), the constraints are first and second class.

Taking Eq. (4.51) into account, one can rewrite the Hamiltonian and the primary constraints, Eqs. (4.49) and (4.48), as

$$H_0 = p_1\dot{q}_1 + p_2\dot{q}_2 + \frac{1}{2}\left(\pi_1^2 + \pi_2^2\right) + \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) \tag{4.52a}$$

$$H_3'^p = p_3 = 0 \tag{4.52b}$$

$$H_3'^\pi = \pi - \dot{q}_3 = 0 \tag{4.52c}$$

Then the corresponding set of HJPDEs, (2.25), reads

$$H_0' = P_0 + H_0 = \frac{\partial S}{\partial t} + \dot{q}_1\frac{\partial S}{\partial q_1} + \dot{q}_2\frac{\partial S}{\partial q_2} + \frac{1}{2}\left(\frac{\partial S}{\partial \dot{q}_1}\right)^2 + \frac{1}{2}\left(\frac{\partial S}{\partial \dot{q}_2}\right)^2 + \frac{1}{2}\dot{q}_1^2 + \frac{1}{2}\dot{q}_2^2 = 0 \tag{4.53a}$$

$$H_3'^p = p_3 = \frac{\partial S}{\partial q_3} = 0 \tag{4.53b}$$

$$H_3'^{\pi} = \pi - \dot{q}_3 = \frac{\partial S}{\partial \dot{q}_3} - \dot{q}_3 = 0 \tag{4.53c}$$

The Hamilton–Jacobi function S can be determined by

$$S(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3, t) = f(t) + W_1(q_1, E_1) + W_2(q_2, E_2) + W_1'(\dot{q}_1, E_1, E_1') + W_2'(\dot{q}_2, E_2, E_2') + f_3(q_3) + f_3'(\dot{q}_3) + A \tag{4.54}$$

The coordinates q_3 and $\dot{q}_3$ are treated as independent variables, just as the time t. Since the Hamiltonian H_0 is time independent, one can write

$$f(t) = -(E_1' + E_2')t$$

Eq. (4.53a) can now be written as

$$-E_1' + \dot{q}_1 \frac{\partial W_1}{\partial q_1} + \frac{1}{2}\left(\frac{\partial W_1'}{\partial \dot{q}_1}\right)^2 + \frac{1}{2}\dot{q}_1^2 - E_2' + \dot{q}_2 \frac{\partial W_2}{\partial q_2} + \frac{1}{2}\left(\frac{\partial W_2'}{\partial \dot{q}_2}\right)^2 + \frac{1}{2}\dot{q}_2^2 = 0 \tag{4.55}$$

From Eq. (4.55) we note that W_1 depends only on q_1 and W_2 depends only on q_2. we can then write

$$\frac{\partial W_1}{\partial q_1} = E_1 \tag{4.56a}$$

so that

$$W_1 = E_1 q_1$$

and

$$\frac{\partial W_2}{\partial q_2} = E_2 \tag{4.56b}$$

so that

$$W_2 = E_2 q_2$$

Substituting Eqs. (4.56), into (4.55), we have

$$-E_1' + \dot{q}_1 E_1 + \frac{1}{2}\left(\frac{\partial W_1'}{\partial \dot{q}_1}\right)^2 + \frac{1}{2}\dot{q}_1^2 - E_2' + \dot{q}_2 E_2 + \frac{1}{2}\left(\frac{\partial W_2'}{\partial \dot{q}_2}\right)^2 + \frac{1}{2}\dot{q}_2^2 = 0 \tag{4.57}$$

Separation of variables in this equation yields

$$\frac{1}{2}\left(\frac{\partial W_1'}{\partial \dot{q}_1}\right)^2 + \frac{1}{2}\dot{q}_1^2 + \dot{q}_1 E_1 - E_1' = 0$$

$$\frac{1}{2}\left(\frac{\partial W_2'}{\partial \dot{q}_2}\right)^2 + \frac{1}{2}\dot{q}_2^2 + \dot{q}_2 E_2 - E_2' = 0$$

The solution of these two equations can be determined as

$$W_1'(\dot{q}_1, E_1, E_1') = \int \sqrt{2E_1' + (E_1)^2 - (\dot{q}_1 + E_1)^2}\, d\dot{q}_1$$

$$W_2'(\dot{q}_2, E_2, E_2') = \int \sqrt{2E_2' + (E_2)^2 - (\dot{q}_2 + E_2)^2}\, d\dot{q}_2$$

Using Eq. (4.53b), one finds $f_3(q_3) = \text{constant}$; and using Eq. (4.53c), one finds $f_3'(\dot{q}_3) = \frac{1}{2}\dot{q}_3^2$.
With these results, the Hamilton–Jacobi function S becomes

$$\begin{aligned} S(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) = {} & (-E_1' - E_2')t + q_1 E_1 + q_2 E_2 \\ & + \int \sqrt{2E_1' + (E_1)^2 - (\dot{q}_1 + E_1)^2}\, d\dot{q}_1 \\ & + \int \sqrt{2E_2' + (E_2)^2 - (\dot{q}_2 + E_2)^2}\, d\dot{q}_2 \\ & + \frac{1}{2}\dot{q}_3^2 + A \end{aligned} \tag{4.58}$$

The solution for the generalized coordinates can be obtained from the transformations (3.12):

$$\eta_1 = \frac{\partial S}{\partial E_1'} = -t + \int \frac{d\dot{q}_1}{\sqrt{2E_1' + (E_1)^2 - (\dot{q}_1 + E_1)^2}} \tag{4.59a}$$

$$\eta_2 = \frac{\partial S}{\partial E_2'} = -t + \int \frac{d\dot{q}_2}{\sqrt{2E_2' + (E_2)^2 - (\dot{q}_2 + E_2)^2}} \tag{4.59b}$$

$$\lambda_1 = \frac{\partial S}{\partial E_1} = q_1 + \int \frac{[E_1 - (\dot{q}_1 + E_1)]\, d\dot{q}_1}{\sqrt{2E_1' + (E_1)^2 - (\dot{q}_1 + E_1)^2}} \tag{4.59c}$$

$$\lambda_2 = \frac{\partial S}{\partial E_2} = q_2 + \int \frac{[E_2 - (\dot{q}_2 + E_2)]\, d\dot{q}_2}{\sqrt{2E_2' + (E_2)^2 - (\dot{q}_2 + E_2)^2}} \tag{4.59d}$$

These four equations can be solved, respectively, to give

$$\dot{q}_1 = \sqrt{2E_1' + (E_1)^2}\, \sin(\eta_1 + t) - E_1 \tag{4.60a}$$

$$\dot{q}_2 = \sqrt{2E_2' + (E_2)^2}\, \sin(\eta_2 + t) - E_2 \tag{4.60b}$$

$$q_1 = \lambda_1 - E_1(\eta_1 + t) - \sqrt{2E_1' + (E_1)^2}\, \cos(\eta_1 + t) \tag{4.60c}$$

$$q_2 = \lambda_2 - E_2(\eta_2 + t) - \sqrt{2E_2' + (E_2)^2}\cos(\eta_2 + t) \tag{4.60d}$$

The other half of the equations of motion can be determined by using Eqs. (3.13), after substituting the results for $\dot{q}_1$ and $\dot{q}_3$:

$$p_1 = \frac{\partial S}{\partial q_1} = E_1 \tag{4.61a}$$

$$p_2 = \frac{\partial S}{\partial q_2} = E_2 \tag{4.61b}$$

$$p_3 = \frac{\partial S}{\partial q_3} = 0 \tag{4.61c}$$

$$\pi_1 = \frac{\partial S}{\partial \dot{q}_1} = \sqrt{2E_1' + (E_1)^2 - (\dot{q}_1 + E_1)^2}$$

$$= \sqrt{2E_1' + (E_1)^2}\,\cos(\eta_1 + t) \tag{4.61d}$$

$$\pi_2 = \frac{\partial S}{\partial \dot{q}_2} = \sqrt{2E_2' + (E_2)^2 - (\dot{q}_2 + E_2)^2}$$

$$= \sqrt{2E_2' + (E_2)^2}\,\cos(\eta_2 + t) \tag{4.61e}$$

$$\pi_3 = \frac{\partial S}{\partial \dot{q}_3} = \dot{q}_3 \tag{4.61f}$$

These results are in exact agreement with those that can be obtained using the canonical approach, Eqs. (2.23), as well as the Dirac approach.

5. CONCLUSION

In this work, a general method for determining the Hamilton–Jacobi function of unconstrained and constrained systems with second-order Lagrangians has been proposed and extended to different kinds of constraints.

We have shown that the Hamilton–Jacobi function S in configuration space can be determined with the proviso that the set of HJPDEs is integrable. The equations of motion can then be readily found using S. These solutions are obtained in terms of the time and the spatial coordinates that correspond to dependent momenta; these are treated as independent variables, just as the time t.

To test our proposed method and to get a somewhat deeper understanding, we have examined one example of discrete regular systems and three examples of different kinds of discrete singular systems. In the first example the results are found to be in exact agreement with the Hamiltonian formalism of regular second-order Lagrangians; while in the examples of singular systems the results are in exact agreement with the canonical approach as well as the Dirac approach.

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